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Koecher-Maaß Series of the Adelic Hermitian Eisenstein Series and the Adelic Hermitian Ikeda Lift for $U(m,m)$

by

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Dedicated to Professor Fumihiro Sato on the occasion of his 65th birthday

Abstract. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$ and with class number h, and χ the Dirichlet character corresponding to the extension $K/\mathbf{Q}$. Let $m = 2n$ or $2n+1$ with n a positive integer. Let f be a primitive form of weight $2k+1$ and character χ for $\Gamma_0(D)$, or a primitive form of weight $2k$ for $SL_2(\mathbf{Z})$ according as $m = 2n$, or $m = 2n+1$. For such an f let $Lift^{(m)}(f)$ be the lift of f to the space of automorphic forms for $U(m,m)$ constructed by Ikeda, and $I_m(f)_i$ be its i-th component for $1 \le i \le h$. First we give an explicit formula for the Koecher-Maass series of $I_m(f)_i$. This gives a refined version of our previous result of [Kat14]. As a consequence, we give an explicit formula for the Koecher-Maass series $L(s, Lift^{(m)}(f))$ of $Lift^{(m)}(f)$. This gives an adelic version of our previous result of [Kat14]. We also give an explicit formula for the Koecher-Maass series of the adelic Hermitian Eisenstein series.

1. Introduction

In [Kat14], we gave an explicit formula of the Koecher-Maass series of the Hermitian Ikeda lift. In this paper, we give a refined version of our previous result, and also give its adelic version. We also give an explicit formula of the Koecher-Maass series of the adelic Hermitian Eisenstein series. Let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$ and with class number h, and Let $\mathcal{O}$ be the ring of integers in K, and χ the Kronecker character corresponding to the extension $K/\mathbf{Q}$. Let k be a non-negative integer. Then for a primitive form $f \in \mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$ and an integer $1 \le i \le h$, Ikeda [Ike08] constructed a lift $I_{2n}(f)_i$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for $\Gamma_i^{(m)}$, where $\Gamma_i^{(m)}$ is a certain Hermitian modular group over K, which will be defined in Section 2. Similarly for a primitive form $f \in \mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ he constructed a lift $I_{2n+1}(f)_i$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for $\Gamma_i^{(m)}$. For the rest of this section, let $m = 2n$ or $m = 2n+1$. We then call $I_m(f)_i$ the Ikeda lift of f for $\Gamma_i^{(m)}$ or the Hermitian Ikeda lift of degree m. Ikeda also showed that the automorphic form $Lift^{(m)}(f)$ on the adele group $\mathcal{U}^{(m)}(\mathbf{A})$ associated

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with $I_m(f)_1, \ldots, I_m(f)_h$ is a cuspidal Hecke eigenform under certain conditions on m and f. We also call $Lift^{(m)}(f)$ the adelic Hermitian Ikeda lift. Then in [Kat14], we expressed the Koecher-Maass series (K-M series) of $I_m(f)_1$ in terms of the L-functions related to f. In this paper we give a similar formula for the Koecher-Maass series of any $I_m(f)_i$ (cf. Theorem 2.1). This gives a refined version of our previous result. As a consequence, we can get an explicit formula of the Koecher-Maass series of $Lift^{(m)}(f)$ (cf. Theorem 2.2). The main tool for giving such formulas is a refined version of the mass formula for the special unitary group of an Hermitian matrix (cf. Proposition 3.6). This enables us to reduce our computation to certain formal power series $P_{m,p}(d;\, X, t)$ in t associated with local Siegel series (cf. Theorem 3.8 and the proof of Theorem 2.1). Then by using results in [Kat14], we give an explicit formula for the Koecher-Maass series of $I_m(f)_i$. We also give another method for giving such a formula for the Koecher-Maass series of $Lift^{(m)}(f)$. Using the same argument as in the proof, we can give an explicit formula of the Koecher-Maass series of the adelic Hermitian Eisenstein series of any degree (cf. Theorem 2.3), which can be regarded as a zeta function of a certain prehomogeneous vector space. We note that explicit formulas of zeta functions of several types of prehomogeneous vector spaces were also given by Saito[Sa99]. The method we use is a refined version of the mass formula for the unitary group of an Hermitian matrix (cf. Proposition 3.1), which is different from the above mass formula. By this formula, we can reduce our computation to a computation of certain formal power series $\widetilde{P}_{m,p}(d;\, X, t)$ in t associated with local Siegel series (cf. Theorem 3.3.) In Section 5, we compute $\widetilde{P}_{m,p}(d;\, X, t)$ using results in [Kat14], and we get an explicit formula for the Koecher-Maass series of $Lift^{(m)}(f)$.

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Notation. Let R be a commutative ring. We denote by $R^\times$ and R^* the semi-group of non-zero elements of R and the unit group of R, respectively. For a subset S of R we denote by $M_{mn}(S)$ the set of (m, n)-matrices with entries in S. In particular put $M_n(S) = M_{nn}(S)$. We denote by 1_m the unit matrix of degree m. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. Let K_0 be a field, and K a quadratic extension of K_0, or $K = K_0 \oplus K_0$. In the latter case, we regard K_0 as a subring of K via the diagonal embedding. We also identify $M_{mn}(K)$ with $M_{mn}(K_0) \oplus M_{mn}(K_0)$ in this case. If K is a quadratic extension of K_0, let ρ be the non-trivial automorphism of K over K_0, and if $K = K_0 \oplus K_0$, let ρ be the automorphism of K defined by $\rho(a, b) = (b, a)$ for $(a, b) \in K$. We sometimes write $\overline{x}$ instead of $\rho(x)$ for $x \in K$ in both cases. Let R be a subring of K. For an (m, n)-matrix $X = (x_{ij})_{m\times n}$ write $X^* = (\overline{x_{ji}})_{n\times m}$, and for an (m, m)-matrix A, we write $A[X] = X^*AX$. Let $\mathrm{Her}_n(R)$ denote the set of Hermitian matrices of degree n with entries in R, that is the subset of $M_n(R)$ consisting of matrices X such that $X^* = X$. Then a Hermitian matrix A of degree n with entries in K is said to be semi-integral over R if $\mathrm{tr}(AB) \in K_0 \cap R$ for any $B \in \mathrm{Her}_n(R)$, where tr denotes the trace of a matrix. We denote by $\widehat{\mathrm{Her}}_n(R)$ the set of semi-integral matrices of degree n over R.

For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. If S is a subset of $\mathrm{Her}_n(\mathbf{C})$ with $\mathbf{C}$ the field of complex numbers, we denote by S^+ the subset of S consisting of positive definite matrices. The group $GL_n(R)$ acts on the set $\mathrm{Her}_n(R)$ in the following way:

$$GL_n(R) \times \mathrm{Her}_n(R) \ni (g, A) \longrightarrow g^* A g \in \mathrm{Her}_n(R)\,.$$

Let G be a subgroup of $GL_n(R)$. For a G-stable subset $\mathcal{B}$ of $\mathrm{Her}_n(R)$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Two Hermitian matrices A and A' with entries in R are said to be G-equivalent and write $A \sim_G A'$ if there is an element X of G such that $A' = A[X]$. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbf{C}$, and for a prime number p we denote by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbf{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbf{Z}[p^{-1}]$.

For a prime number p we denote by $\mathrm{ord}_p(*)$ the additive valuation of $\mathbf{Q}_p$ normalized so that $\mathrm{ord}_p(p) = 1$, and put $|x|_p = p^{-\mathrm{ord}_p(x)}$. Moreover we denote by $|x|_\infty$ the absolute value of $x \in \mathbf{C}$. For a prime number p put $K_p = K \otimes \mathbf{Q}_p$, and $\mathcal{O}_p = \mathcal{O} \otimes \mathbf{Z}_p$. Then K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. In the former case, for $x \in K_p$, we denote by $\overline{x}$ the conjugate of x over $\mathbf{Q}_p$. In the latter case, for $x = (x_1, x_2)$ with $x_i \in \mathbf{Q}_p$, we put $\overline{x} = (x_2, x_1)$. For $x \in K_p$ we define the norm $N_{K_p/\mathbf{Q}_p}(x)$ by $N_{K_p/\mathbf{Q}_p}(x) = x\overline{x}$, and put $\nu_{K_p}(x) = \mathrm{ord}_p(N_{K_p/\mathbf{Q}_p}(x))$, and $|x|_{K_p} = |N_{K_p/\mathbf{Q}_p}(x)|_p$. Moreover put $|x|_{K_\infty} = |x\overline{x}|_\infty$ for $x \in \mathbf{C}$.

2. Main results

For a positive integer N let $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \bmod N\}$, and for a Dirichlet character ψ mod N, we denote by $\mathfrak{M}_l(\Gamma_0(N), \psi)$ the space of modular forms of weight l for $\Gamma_0(N)$ and nebentype ψ, and by $\mathfrak{S}_l(\Gamma_0(N), \psi)$ its subspace consisting of cusp forms. We simply write $\mathfrak{M}_l(\Gamma_0(N), \psi)$ (resp. $\mathfrak{S}_l(\Gamma_0(N), \psi)$) as $\mathfrak{M}_l(\Gamma_0(N))$ (resp. as $\mathfrak{S}_l(\Gamma_0(N))$) if ψ is the trivial character.

Throughout the paper, we fix an imaginary quadratic extension K of $\mathbf{Q}$ with discriminant $-D$, and denote by $\mathcal{O}$ the ring of integers in K. For a non-degenerate Hermitian matrix or alternating matrix T with entries in K, let $\mathcal{U}_T$ be the unitary group defined over $\mathbf{Q}$, whose group $\mathcal{U}_T(R)$ of R-valued points is given by

$$\mathcal{U}_T(R) = \{g \in GL_m(R \otimes K) \mid T[g] = T\}$$

for any $\mathbf{Q}$-algebra R, where $\overline{g}$ denotes the automorphism of $M_n(R \otimes K)$ induced by the non-trivial automorphism of K over $\mathbf{Q}$. We also define the special unitary group $\mathcal{SU}_T$ over $\mathbf{Q}$ by $\mathcal{SU}_T = \mathcal{U}_T \cap \mathrm{Res}_{K/\mathbf{Q}}(SL_m)$, where $\mathrm{Res}_{K/\mathbf{Q}}$ is the Weil restriction. In particular we write $\mathcal{U}_{J_m}$ as $\mathcal{U}^{(m)}$ or $U(m, m)$, where $J_m = \begin{pmatrix} O & -1_m \\ 1_m & O \end{pmatrix}$. Then

$$\mathcal{U}^{(m)}(\mathbf{Q}) = \{M \in GL_{2m}(K) \mid J_m[M] = J_m\}\,.$$

Put

$$\Gamma^{(m)} = \Gamma_K^{(m)} = \mathcal{U}^{(m)}(\mathbf{Q}) \cap GL_{2m}(\mathcal{O}).$$

Let $\mathfrak{H}_m$ be the Hermitian upper half-space defined by

$$\mathfrak{H}_m = \{Z \in M_m(\mathbf{C}) \mid \frac{1}{2\sqrt{-1}}(Z - Z^*) \text{ is positive definite}\}.$$

The group $\mathcal{U}^{(m)}(\mathbf{R})$ acts on $\mathfrak{H}_m$ by

$$g\langle Z \rangle = (AZ + B)(CZ + D)^{-1} \text{ for } g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{U}^{(m)}(\mathbf{R}),\ Z \in \mathfrak{H}_m.$$

We also put $j(g, Z) = CZ + D$ for such Z and g. Let l be an integer. For a subgroup Γ of $\mathcal{U}^{(m)}(\mathbf{Q})$ commensurable with $\Gamma^{(m)}$ and a character ψ of Γ, we denote by $\mathfrak{M}_l(\Gamma, \psi)$ the space of holomorphic modular forms of weight l with character ψ for Γ. We denote by $\mathfrak{S}_l(\Gamma, \psi)$ the subspace of $\mathfrak{M}_l(\Gamma, \psi)$ consisting of cusp forms. In particular, if ψ is the character of Γ defined by $\psi(\gamma) = (\det \gamma)^{-l}$ for $\gamma \in \Gamma$, we write $\mathfrak{M}_{2l}(\Gamma, \psi)$ as $\mathfrak{M}_{2l}(\Gamma, \det^{-l})$, and so on.

Now we consider the adelic modular form. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$, and $\mathbf{A}_f$ the non-archimedean factor of $\mathbf{A}$. Let $h = h_K$ be a class number of K. Let $G^{(m)} = \mathrm{Res}_{K/\mathbf{Q}}(GL_m)$, and $G^{(m)}(\mathbf{A})$ be the adelization of $G^{(m)}$. Moreover put $\mathcal{C}^{(m)} = \prod_p GL_m(\mathcal{O}_p)$. Let $\mathcal{U}^{(m)}(\mathbf{A})$ be the adelization of $\mathcal{U}^{(m)}$. We define the compact subgroup $\mathcal{K}_0^{(m)}$ of $\mathcal{U}^{(m)}(\mathbf{A}_f)$ by $\mathcal{U}^{(m)}(\mathbf{A}) \cap \prod_p GL_{2m}(\mathcal{O}_p)$, where p runs over all rational primes. Then we have

$$\mathcal{U}^{(m)}(\mathbf{A}) = \bigsqcup_{i=1}^{h} \mathcal{U}^{(m)}(\mathbf{Q})\gamma_i \mathcal{K}_0^{(m)} \mathcal{U}^{(m)}(\mathbf{R})$$

with some subset $\{\gamma_1, ..., \gamma_h\}$ of $\mathcal{U}^{(m)}(\mathbf{A}_f)$. We can take γ_i as

$$\gamma_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{*-1} \end{pmatrix},$$

where $\{t_i\}_{i=1}^h = \{(t_{i,p})\}_{i=1}^h$ is a certain subset of $G^{(m)}(\mathbf{A}_f)$ such that $t_1 = 1_m$, and

$$G^{(m)}(\mathbf{A}) = \bigsqcup_{i=1}^{h} G^{(m)}(\mathbf{Q}) t_i G^{(m)}(\mathbf{R}) \mathcal{C}^{(m)}.$$

Put $\Gamma_i^{(m)} = \mathcal{U}^{(m)}(\mathbf{Q}) \cap \gamma_i \mathcal{K}_0^{(m)} \gamma_i^{-1} \mathcal{U}^{(m)}(\mathbf{R})$. Then for an element $(F_1, ..., F_h) \in \bigoplus_{i=1}^h \mathfrak{M}_{2l}(\Gamma_i^{(m)}, \det^{-l})$, we define $(F_1, ..., F_h)^\sharp$ by

$$(F_1, ..., F_h)^\sharp(g) = F_i(x\langle \mathbf{i} \rangle) j(x, \mathbf{i})^{-2l} (\det x)^l$$

for $g = u\gamma_i x\kappa$ with $u \in \mathcal{U}^{(m)}(\mathbf{Q})$, $x \in \mathcal{U}^{(m)}(\mathbf{R})$, $\kappa \in \mathcal{K}_0^{(m)}$, where $\mathbf{i} = \sqrt{-1}1_m$. We denote by $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$ the space of automorphic forms obtained in this way. We also put

$$\mathcal{S}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l}) = \{(F_1, ..., F_h)^\sharp \mid F_i \in \mathfrak{S}_{2l}(\Gamma_i^{(m)}, \det^{-l})\}.$$

We can define the Hecke operators which act on the space
$\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$. For the precise definition of them, see [Ike08].

Now let $F(Z) \in \mathfrak{M}_{2l}(\Gamma_i^{(m)}, \det^{-l})$. Then $F_i(Z)$ has the following Fourier expansion:

$$F(Z) = \sum_{\substack{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})_i, \\ \text{T :semi positive definite}}} a_F(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$\widehat{\mathrm{Her}}_m(\mathcal{O})_i = \{T \in \mathrm{Her}_m(K) \mid t_{i,p}^* T t_{i,p} \in \widehat{\mathrm{Her}}_m(\mathcal{O}_p) \text{ for any } p\}.$$

We then define the Koecher-Maass series $L^*(s, F)$ and $L(s, F)$ of F by

$$L^*(s, F) = \prod_p |\det t_{i,p}|_{K_p}^{s-l} \sum_{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})_i{}^+/U_i} \frac{a_F(T)}{(\det T)^s \#(U_{i,T})},$$

and

$$L(s, F) = \prod_p |\det t_{i,p}|_{K_p}^{s-l} \sum_{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})_i{}^+/U_i'} \frac{a_F(T)}{(\det T)^s \#(U_{i,T}')},$$

where $U_i = GL_m(K) \cap (GL_m(\mathbf{C})t_i \prod_p GL_m(\mathcal{O}_p)t_i^{-1})$, $U_{i,T} = \{\gamma \in U_i \mid \gamma^* T \gamma = T\}$, and $U_i' = U_i \cap SL_m(K)$, $U_{i,T}' = U_{i,T} \cap SL_m(K)$. We also define the Koecher-Maass series $L^*(s, F)$ of $F = (F_1, \ldots, F_h)^\# \in \mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$ by

$$L^*(s, F) = \sum_{i=1}^h L^*(s, F_i).$$

We note that

$$L^*(s, F_i) = \frac{1}{\#(\mathcal{O}^*)} L(s, F_i),$$

and hence

$$L^*(s, F) = \frac{1}{\#(\mathcal{O}^*)} \sum_{i=1}^h L(s, F_i).$$

For a non-degenerate Hermitian matrix B with entries in K_p of degree m, put $\gamma(B) = (-D)^{[m/2]} \det B$. We put $\xi_p = 1, -1$, or 0 according as $K_p \cong \mathbf{Q}_p \oplus \mathbf{Q}_p$, K_p is an unramified quadratic extension of $\mathbf{Q}_p$, or K_p is a ramified quadratic extension of $\mathbf{Q}_p$. For $T \in \widehat{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ we define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in \mathrm{Her}_n(K_p)/\mathrm{Her}_n(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(TR)) p^{-\mathrm{ord}_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathcal{O}_p^m + \mathcal{O}_p^m : \mathcal{O}_p^m]$. We remark that there exists a unique polynomial $F_p(T, X)$ in X such that

$$b_p(T, s) = F_p(T, p^{-s}) \prod_{i=0}^{[(m-1)/2]} (1 - p^{2i-s}) \prod_{i=1}^{[m/2]} (1 - \xi_p p^{2i-1-s})$$

(cf. Shimura [Sh97]). We then define a Laurent polynomial $\widetilde{F}_p(T, X)$ as

$$\widetilde{F}_p(T, X) = X^{-\mathrm{ord}_p(\gamma(T))} F_p(T, p^{-m} X^2).$$

We remark that we have

$$\widetilde{F}_p(T, X^{-1}) = (-D, \gamma(T))_p \widetilde{F}_p(T, X) \qquad \text{if } m \text{ is even,}$$

$$\widetilde{F}_p(T, \xi_p X^{-1}) = \widetilde{F}_p(T, X) \qquad \text{if } m \text{ is even and } p \nmid D,$$

and

$$\widetilde{F}_p(T, X^{-1}) = \widetilde{F}_p(T, X) \qquad \text{if } m \text{ is odd}$$

(cf. [Ike08]). Here $(a, b)_p$ is the Hilbert symbol of $a, b \in \mathbf{Q}_p^\times$. Hence we have

$$\widetilde{F}_p(T, X) = (-D, \gamma(B))_p^{m-1} X^{\mathrm{ord}_p(\gamma(T))} F_p(T, p^{-m} X^{-2}).$$

First let k be a non-negative integer, and $m = 2n$ a positive even integer. Let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. For a prime number p not dividing D let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \chi(p)\alpha_p^{-1} = p^{-k}a(p)$, and for $p \mid D$ put $\alpha_p = p^{-k}a(p)$. We note that $\alpha_p \neq 0$ even if $p|D$. Then for the Kronecker character χ we define Hecke's L-function $L(s, f, \chi^i)$ twisted by χ^i as

$$L(s, f, \chi^i) = \prod_{p \nmid D} \{(1 - \alpha_p p^{-s+k}\chi(p)^i)(1 - \alpha_p^{-1} p^{-s+k}\chi(p)^{i+1})\}^{-1}$$

$$\times \begin{cases} \prod_{p|D}(1 - \alpha_p p^{-s+k})^{-1} & \text{if } i \text{ is even} \\ \prod_{p|D}(1 - \alpha_p^{-1} p^{-s+k})^{-1} & \text{if } i \text{ is odd.} \end{cases}$$

In particular, if i is even, we sometimes write $L(s, f, \chi^i)$ as $L(s, f)$ as usual. Moreover for $i = 1, \ldots, h$ we define a Fourier series

$$I_m(f)_i(Z) = \sum_{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})_i^+} a_{I_m(f)_i}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_{2n}(f)_i}(T) = |\gamma(T)|^k \prod_p |\det(t_{i,p})|_{K_p}^n \widetilde{F}_p(t_{i,p}^* T t_{i,p}, \alpha_p^{-1}).$$

Next let k be a positive integer and $m = 2n + 1$ a positive odd integer. Let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. For a prime number p let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+1/2}a(p)$. Then we define Hecke's L-function $L(s, f, \chi^i)$ twisted by χ^i as

$$L(s, f, \chi^i) = \prod_p \{(1 - \alpha_p p^{-s+k-1/2}\chi(p)^i)(1 - \alpha_p^{-1} p^{-s+k-1/2}\chi(p)^i)\}^{-1}.$$

In particular, if i is even we write $L(s, f, \chi^i)$ as $L(s, f)$ as usual. Moreover for $i = 1, ..., h$ we define a Fourier series

$$I_{2n+1}(f)_i(Z) = \sum_{T \in \widehat{\mathrm{Her}}_{2n+1}(\mathcal{O})_i^+} a_{I_{2n+1}(f)_i}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_{2n+1}(f)_i}(T) = |\gamma(T)|^{k-1/2} \prod_p |\det(t_{i,p})|_{K_p}^{n+1/2} \widetilde{F}_p(t_{i,p}^* T t_{i,p}, \alpha_p^{-1}) .$$

Remark. In [Ike08], Ikeda defined $\widetilde{F}_p(T, X)$ as

$$\widetilde{F}_p(T, X) = X^{\mathrm{ord}_p(\gamma(T))} F_p(T, p^{-m} X^{-2}) ,$$

and we define it by replacing X with X^{-1} in this paper. This change does not affect the results.

Then Ikeda [Ike08] showed the following:

Let $m = 2n$ or $2n + 1$. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $m = 2n + 1$. Then $I_m(f)_i(Z)$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma_i^{(m)}, \det^{-k-n})$ for any i. In particular, $I_m(f) := I_m(f)_1$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma^{(m)}, \det^{-k-n})$.

This is a Hermitian analogue of the lifting constructed in [Ike01]. We call $I_m(f)_i$ the Ikeda lift of f for $\Gamma_i^{(m)}$. By the above result, we can define an element $(I_m(f)_1, ..., I_m(f)_h)^\sharp$ of $\mathcal{S}_{2k+2n}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-k-n})$, which we write $Lift^{(m)}(f)$. Then Ikeda also showed the following

Let $m = 2n$ or $2n + 1$. Suppose that $Lift^{(m)}(f)$ is not identically zero. Then $Lift^{(m)}(f)$ is a Hecke eigenform in $\mathcal{S}_{2k+2n}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}), \det^{-k-n})$ and its standard L-function $L(s, Lift^{(m)}(f), \mathrm{st})$ coincides with

$$\prod_{i=1}^{m} L(s + k + n - i + 1/2, f)L(s + k + n - i + 1/2, f, \chi)$$

up to bad Euler factors.

We call $Lift^{(m)}(f)$ the adelic Ikeda lift of f for $\mathcal{U}^{(m)}$.

Let Q_D be the set of prime divisors of D. For each prime $q \in Q_D$, put $D_q = q^{\mathrm{ord}_q(D)}$. We define a Dirichlet character χ_q by

$$\chi_q(a) = \begin{cases} \chi(a') & \text{if } (a, q) = 1 \\ 0 & \text{if } q|a \end{cases} ,$$

where a' is an integer such that

$$a' \equiv a \bmod D_q \quad \text{ and } a' \equiv 1 \bmod DD_q^{-1} .$$

For a subset Q of Q_D put $\chi_Q = \prod_{q \in Q} \chi_q$ and $\chi'_Q = \prod_{q \in Q_D, q \notin Q} \chi_q$. Here we make the convention that $\chi_Q = 1$ and $\chi'_Q = \chi$ if Q is the empty set. Let

$$f(z) = \sum_{N=1}^{\infty} c_f(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. Then there exists a primitive form

$$f_Q(z) = \sum_{N=1}^{\infty} c_{f_Q}(N)\mathbf{e}(Nz)$$

such that

$$c_{f_Q}(p) = \begin{cases} \chi_Q(p)c_f(p) & \text{if } p \notin Q \\ \\ \chi'_Q(p)\overline{c_f(p)} & \text{if } p \in Q \end{cases}.$$

We define $L(s, \chi^i) = \zeta(s)$ or $L(s, \chi)$ according as i is even or odd, where $\zeta(s)$ and $L(s, \chi)$ are Riemann's zeta function, and, the Dirichlet L-function for χ, respectively. Moreover we put

$$\widetilde{\Lambda}(s, \chi^i) = 2(2\pi)^{-s}\Gamma(s)L(s, \chi^i)$$

with $\Gamma(s)$ the Gamma function. Then our main results in this paper are as follows:

THEOREM 2.1. *Let r be a positive integer not greater than h.*
(1) *Put $c_r = \prod_p p^{-\nu_{K_p}(\det t_{r,p})}$. Let k be a nonnegative integer and n a positive integer. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. Then, we have*

$$L(s, I_{2n}(f)_r) = D^{ns+n^2-n/2-1/2}2^{-2n+1}$$
$$\times \prod_{i=2}^{2n} \widetilde{\Lambda}(i, \chi^i) \sum_{Q \in \mathcal{Q}_D} \chi_Q((-1)^n c_r) \prod_{i=1}^{2n} L(s-2n+i, f_Q, \chi^{i-1}).$$

(2) *Let k be a positive integer and n a non-negative integer. Let f be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Then, we have*

$$L(s, I_{2n+1}(f)_r) = D^{ns+n^2+3n/2}2^{-2n} \prod_{i=2}^{2n+1} \widetilde{\Lambda}(i, \chi^i) \prod_{i=1}^{2n+1} L(s-2n-1+i, f, \chi^{i-1}).$$

In the case $r = 1$, the above results coincide with [[Kat14], Theorems 2.3 and 2.4]. We will prove the above theorem in Section 4. By the genus theory of quadratic fields, we have

$$\sum_{r=1}^{h} \chi_Q((-1)^n c_r) = h,\ \ \chi((-1)^n)h,\ \text{ or } 0$$

according as $Q = \emptyset, Q_D$, or otherwise. Moreover we have $\widetilde{\Lambda}(1, \chi)D^{1/2}\#(\mathcal{O}^*)/2 = h$. Hence we obtain

THEOREM 2.2. (1) *Let k be a nonnegative integer and n a positive integer. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. Then, we have*

$$L^*(s, Lift^{(2n)}(f)) = D^{ns+n^2-n/2}2^{-2n} \prod_{i=1}^{2n} \widetilde{\Lambda}(i, \chi^i)$$
$$\times \{\prod_{i=1}^{2n} L(s-2n+i, f, \chi^{i-1}) + \chi((-1)^n) \prod_{i=1}^{2n} L(s-2n+i, f, \chi^{i})\}.$$

(2) *Let k be a positive integer and n a non-negative integer. Let f be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Then, we have*

$$L^*(s, Lift^{(2n+1)}(f)) = D^{ns+n^2+3n/2+1/2}2^{-2n-1}\prod_{i=1}^{2n+1}\widetilde{\Lambda}(i,\chi^i)\prod_{i=1}^{2n+1}L(s-2n-1+i, f, \chi^{i-1}) .$$

In the case where m is odd, $L^*(s, Lift^{(2n+1)}(f))$ coincides with $L(s, I_m(f))$ up to constant multiple, however, in the case where m is even, it does not, and it becomes simpler than $L(s, I_m(f))$ in general. We will give another proof to Theorem 2.2 in Section 5. By using the same argument as in Section 5, we can also give an explicit formula for the Koecher-Maass series of the adelic Eisenstein series on $\mathcal{U}^{(m)}(\mathbf{A})$. Let $\mathcal{P}$ be the maximal parabolic subgroup of $\mathcal{U}^{(m)}$ defined by

$$\mathcal{P}(R) = \{\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathcal{U}^{(m)}(R)\}$$

for any $\mathbf{Q}$-algebra R. Write an element $g = (g_v) \in \mathcal{U}^{(m)}(\mathbf{A})$ as

$$(g_p)_{p<\infty} = \left(\left(\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}\right)\right)_{p<\infty} (\kappa_p)_{p<\infty}$$

with $\left(\left(\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}\right)\right)_{p<\infty} \in \prod_{p<\infty}\mathcal{P}(\mathbf{Q}_p)$ and $(\kappa_p)_{p<\infty} \in \mathcal{K}_0^{(m)}$, and define the function on $\mathcal{U}^{(m)}(\mathbf{A})$ by

$$\mathbf{f}_{2l}(g) = \prod_p |\det(d_p\overline{d}_p)|_p^{-l} j(g_\infty, \mathbf{i})^{-2l}(\det g_\infty)^l .$$

We then define the normalized Eisenstein series as

$$\mathbf{E}_{2l}^{(m)}(g) = 2^{-m}\prod_{i=1}^m L(i-2l, \chi^{i-1}) \sum_{\gamma\in\mathcal{P}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{Q})} \mathbf{f}_{2l}(\gamma g) .$$

Let $t_i = (t_{i,p})$ be the element of $G^{(m)}(\mathbf{A}_f)$ defined before. Then we note that $\mathbf{E}_{2l}^{(m)}$ is written as

$$\mathbf{E}_{2l}^{(m)} = (\mathcal{E}_{2l,1}^{(m)}, \mathcal{E}_{2l,2}^{(m)}, ..., \mathcal{E}_{2l,h}^{(m)})^\sharp ,$$

where

$$\mathcal{E}_{2l,i}^{(m)}(Z)$$
$$= 2^{-m}\prod_p |\det(t_{i,p})|_{K_p}^l \prod_{j=1}^m L(j-2l, \chi^{j-1}) \sum_{g\in(\Gamma_i^{(m)}\cap\mathcal{P}(\mathbf{Q}))\backslash\Gamma_i^{(m)}} (\det g)^l j(g, Z)^{-2l}$$

for $i = 1, ..., h$. We note that for $T \in \widehat{\mathrm{Her}}_m(\mathcal{O})_i^+$, the T-th Fourier coefficient of $\mathcal{E}_{2l,i}^{(m)}(Z)$ is given by

$$|\gamma(T)|^{l-m/2}\prod_p |\det(t_{i,p})|_{K_p}^{m/2}\widetilde{F}_p(t_{i,p}^* T t_{i,p}, p^{l-m/2}) .$$

THEOREM 2.3. (1) *Let k be a nonnegative integer and n a positive integer. Then, we have*

$$L^*(s, \mathbf{E}_{2k+2n}^{(2n)}) = D^{ns+n^2-n/2}2^{-2n}\prod_{i=1}^{2n}\widetilde{\Lambda}(i, \chi^i)$$

$$\times\{\prod_{i=1}^{2n} L(s-2n+i,\chi^{i-1})L(s-2k-2n+i,\chi^{i})$$

$$+\chi((-1)^n)\prod_{i=1}^{2n} L(s-2n+i,\chi^{i})L(s-2k-2n+i,\chi^{i-1}\}.$$

(2) *Let k be a positive integer and n a non-negative integer. Let f be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Then, we have*

$$L^*(s,\mathbf{E}_{2k+2n}^{(2n+1)}) = D^{ns+n^2+3n/2+1/2}2^{-2n-1}\prod_{i=1}^{2n+1}\widetilde{\Lambda}(i,\chi^i)$$

$$\times\prod_{i=1}^{2n+1} L(s-2n-1+i,\chi^{i-1})L(s-2k-2n-1+i,\chi^{i}).$$

3. Reduction to local computations

To prove our main results, we give explicit formulas for $R(s, I_m(f)_i)$. To do this, we reduce the problem to local computations. Throughout the rest of this paper, let K_p be a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. In the former case let $\mathcal{O}_p$ be the ring of integers in K_p, and f_p the exponent of the conductor of $K_p/\mathbf{Q}_p$, and put $e_p = f_p - \delta_{2,p}$, where $\delta_{2,p}$ is Kronecker's delta. In the latter case, put $\mathcal{O}_p = \mathbf{Z}_p \oplus \mathbf{Z}_p$, and $e_p = f_p = 0$. Moreover put $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = p^{e_p}\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$. We note that $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = \mathrm{Her}_m(\mathcal{O}_p)$ if K_p is not ramified over $\mathbf{Q}_p$. Let K be an imaginary quadratic extension of $\mathbf{Q}$ with the discriminant $-D$. We then put $\widetilde{D} = \prod_{p|D} p^{e_p}$, and $\widetilde{\mathrm{Her}}_m(\mathcal{O}) = \widetilde{D}\mathrm{Her}_m(\mathcal{O})$. Now let m and l be positive integers such that $m \geq l$. Then for an integer a and $A \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, $B \in \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)$ put

$$\mathcal{A}_a(A,B) = \{X \in M_{ml}(\mathcal{O}_p)/p^a M_{ml}(\mathcal{O}_p) \mid A[X]-B \in p^a\widetilde{\mathrm{Her}}_l(\mathcal{O}_p)\},$$

and

$$\mathcal{B}_a(A,B) = \{X \in \mathcal{A}_a(A,B) \mid \mathrm{rank}_{\mathcal{O}_p/p\mathcal{O}_p} X = l\}.$$

Assume that A and B are non-degenerate. Then the number $p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B)$ is independent of a if a is sufficiently large. Hence we define the local density $\alpha_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B).$$

In particular we write $\alpha_p(A) = \alpha_p(A,A)$. For an element $A \in \mathrm{Her}_m(\mathcal{O}_p)^\times$ put

$$\upsilon_p(A) = \lim_{a\to\infty} p^{-am^2}\#(\Upsilon_a(A)),$$

where

$$\Upsilon_a(A) = \{X \in M_{ml}(\mathcal{O}_p)/p^a M_{ml}(\mathcal{O}_p) \mid A[X]-A \in p^a\mathrm{Her}_l(\mathcal{O}_p)\}.$$

Let $\{t_i\}$ be elements of $G^{(m)}(\mathbf{A}_f)$ defined as before. For each i, put

$$\widetilde{\mathrm{Her}}_m(\mathcal{O})_i^+ = \prod_p (t_{i,p}^{*}{}^{-1}\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)t_{i,p}^{-1}) \cap \mathrm{Her}_m(K)^+ .$$

Fix a positive integer $r \le h$. For $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})_r^+$ and a positive integer $i \le h$, let $\mathcal{G}_{r,i}(T)$ denote the set of U_i-equivalence classes of elements T' of $\mathrm{Her}_m(K)^+$ such that $t_i^* T' t_i$ belongs to the genus of $t_r^* T t_r$. We fix a complete set of representatives of $\mathcal{G}_{r,i}(T)$ and denote it by the same symbol $\mathcal{G}_{r,i}(T)$. Furthermore put

$$M_r(T) = \sum_{i=1}^{h} \sum_{T' \in \mathcal{G}r,i(T)} \frac{1}{\#(U_{i,T'})} .$$

PROPOSITION 3.1. *Let the notation be as above. Then*

$$M_r(T) = \frac{2^{c_D m^2} \kappa_m \det T^m \prod_p p^{\nu_{K_p}(\det t_{r,p})m}}{D^{m(m+1)/4} \prod_p \alpha_p(t_{r,p}^* T t_{r,p})}$$

where $c_D = 1$ *or* 0 *according as* 2 *divides* D *or not, and*

$$\kappa_m = 2^{-m+1} \prod_{j=1}^{m} \Gamma_{\mathbf{C}}(j) .$$

Proof. Let M_m be the affine space of all $m \times m$ matrices defined over K, and $\mathrm{Her}_m = \{X \in M_m \mid X^* = X\}$, We take an element η of $\mathcal{O}$ such that $\{1, \eta\}$ forms an integral basis of $\mathcal{O}$. We then define an m^2 form ω on $\mathcal{U}_T$ by

$$\omega(z) = (-\sqrt{-D})^{-m(m+1)} d\lambda_m(z)/d\sigma_m(z^* T z) ,$$

where

$$d\lambda_m(z) = \wedge_{1\le i,j\le m}(dz_{ij} \wedge d\overline{z}_{ij}), \quad z = (z_{ij}) = (x_{ij}) + (y_{ij})\eta \in \mathrm{Res}_{K/\mathbf{Q}}(M_m) ,$$

and

$$d\sigma_m(s) = \wedge_{i=1}^m ds_{ii} \wedge \wedge_{1\le i<j\le m}(ds_{ij} \wedge d\overline{s}_{ij}), \quad s = (s_{ij}) = (u_{ij}) + (v_{ij})\eta \in \mathrm{Res}_{K/\mathbf{Q}}(\mathrm{Her}_m) .$$

Since $d\lambda_m(z)(\det(z_{ij})\overline{\det(z_{ij})})^{-m}$ and $d\sigma_m(z^*Tz)(\det(z_{ij})\overline{\det(z_{ij})})^{-m}$ are $\mathbf{G}^{(m)}$-invariant forms, ω defines a gauge form on $\mathcal{U}_T$. We note that

$$d\lambda_m(z) = (-\sqrt{-D})^{2m^2} \wedge_{1\le i,j\le m} (dx_{ij} \wedge dy_{ij}) ,$$

and

$$d\sigma_m(z^*Tz) = (-\sqrt{-D})^{-m(m-1)} \wedge_{i=1}^m df_{ii} \wedge \wedge_{1\le i<j\le m}(dg_{ij} \wedge dh_{ij}) ,$$

where $z^*Tz = (f_{ij})_{m\times m}$ with $f_{ij} = g_{ij} + h_{ij}\eta$ for $i \ne j$. Hence we have

$$\omega(z) = \wedge_{1\le i,j\le m}(dx_{ij} \wedge dy_{ij})/(\wedge_{i=1}^m df_{ii} \wedge \wedge_{1\le i<j\le m}(dg_{ij} \wedge dh_{ij})).$$

Put

$$\widetilde{u}_p = \begin{cases} (1+p^{-1})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ (1-p^{-1})^{-1} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ 1 & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

Then we can take $\{\widetilde{u}_p^{-1}\}_p$ as the convergence factors of the Tamagawa measure $|\omega|_{\mathbf{A}}$ and

$$|\omega|_{\mathbf{A}} = L(1,\chi)^{-1}|\omega|_\infty \prod_p (\widetilde{u}_p^{-1}|\omega|_p)\,.$$

We have the following decomposition of $\mathcal{U}_T(\mathbf{A})$:

$$\mathcal{U}_T(\mathbf{A}) = \bigsqcup_{i=1}^{H} \mathcal{U}_T(\mathbf{Q})\delta_j \mathcal{K}_{r,T}\,,$$

where $\mathcal{K}_{r,T} = \mathcal{U}_T(\mathbf{R})\prod_{p<\infty}(t_{r,p}GL_m(\mathcal{O}_p)t_{r,p}^{-1} \cap \mathcal{U}_T(K_p))$, and H is the class number of $\mathcal{U}_T$ with respect to $\mathcal{K}_{r,T}$. It is known that the Tamagawa number of $\mathcal{U}_T$ is 2 (cf. [Mar69]). Hence by using the standard method, we can prove that

$$L(1,\chi)^{-1}\sum_{j=1}^{H}\#(\mathfrak{G}_j)^{-1}\int_{\mathcal{U}_T(\mathbf{R})}|\omega|_\infty \prod_{p<\infty}(\widetilde{u}_p^{-1}\int_{t_{r,p}GL_m(\mathcal{O}_p)t_{r,p}^{-1}\cap\mathcal{U}_T(K_p)}|\omega|_p) = 2\,,$$

where

$$\mathfrak{G}_j = \delta_j\mathcal{K}_{r,T}\delta_j^{-1}\cap\mathcal{U}_T(\mathbf{Q})\,.$$

Let $\mathcal{R} = \{\mathfrak{G}_j\}_{j=1}^H$, and put $\mathcal{R}' = \bigsqcup_{i=1}^h \mathcal{R}'_i$, where $\mathcal{R}'_i = \{U_{i,T'} \mid T' \in \mathcal{G}_{r,i}(T)\}$. Take an element $\mathfrak{G}_j = \delta_j\mathcal{K}_{r,T}\delta_j^{-1}\cap\mathcal{U}_T(\mathbf{Q})$ of $\mathcal{R}$. Then $\delta_j t_r$ can be written as $\delta_j t_r = \widetilde{a}t_{i_j}\widetilde{v}$ with some integer $1 \le i_j \le h$ and $\widetilde{a} \in \mathbf{G}^{(m)}(\mathbf{Q})$ and $\widetilde{v} \in \mathcal{C}^{(m)}\mathbf{G}^{(m)}(\mathbf{R})$. Put $T'' = \widetilde{a}^*T\widetilde{a}$. Then we easily see that T'' is an element of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)_{i_j}^+$, and $t_{i_j}^*T''t_{i_j}$ belongs the genus of $t_r^*Tt_r$. Thus there exists an element $u \in U_{i_j}$ such that $u^*T''u \in \mathcal{G}_{r,i_j}(T)$. Hence we can take an element $a \in \mathbf{G}^{(m)}(\mathbf{Q})$ and $v \in \mathcal{C}^{(m)}\mathbf{G}^{(m)}(\mathbf{R})$ such that $\delta_j t_r = at_{i_j}v$ and $a^{-1}\mathfrak{G}_j a = U_{i_j,T'}$ for some $T' \in \mathcal{G}_{r,i}(T)$. Hence we can define a mapping Φ from $\mathcal{R}$ to $\mathcal{R}'$ by $\Phi(\mathfrak{G}_j) = a^{-1}\mathfrak{G}_j a$. It is easily seen that Φ is bijective. Hence

$$\sum_{j=1}^{H}\#(\mathfrak{G}_j)^{-1} = \sum_{i=1}^{h}\sum_{T'\in\mathcal{G}_{r,i}(T)}\#(U_{i,T'})^{-1}\,.$$

On the other hand we have

$$\int_{t_{r,p}GL_m(\mathcal{O}_p)t_{r,p}^{-1}\cap\mathcal{U}_T(K_p)}|\omega|_p = p^{-\nu_{K_p}(\det t_{r,p})m}\int_{X_{r,T}(\mathcal{O}_p)}|\omega_{r,p}|\,,$$

where $\omega_{r,p}$ is the gauge form on $\mathcal{U}_T(K_p)$ defined by

$$\omega_{r,p}(x) = (-\sqrt{-D})^{-m(m+1)}d\lambda_m(x)/d\sigma_m(x^*t_{r,p}^*Tt_{r,p}x),$$

and

$$X_{r,T}(\mathcal{O}_p) = \{X \in M_m(\mathcal{O}_p) \mid t_{r,p}^*Tt_{r,p}[X] = t_{r,p}^*Tt_{r,p}\}\,.$$

By the definition of the measure $|\omega_{r,p}|$ and [Ya83], we have

$$\int_{X_{r,T}(\mathcal{O}_p)}|\omega_{r,p}| = \upsilon_p(t_{r,p}^*Tt_{r_p})\,.$$

Thus by [Kat14], Lemma 3.1, we have

$$\int_{X_{r,T}(\mathcal{O}_p)} |\omega_{r,p}| = p^{-\nu_{K_p}(\det t_{r,p})m} p^{m(m+1)f_p/2-m^2\delta_{2,p}} \alpha_p(t_{r,p}^* T t_{r,p}) ,$$

where $\delta_{2,p}$ is Kronecker's delta. On the other hand,

$$\int_{\mathcal{U}_T(\mathbf{R})} |\omega|_\infty = \left(\frac{\det T^m D^{m(m+1)/4} \Gamma_{\mathbf{C}}(i)}{2^m} \right)^{-1} .$$

We note that the infinite product $\prod_p \alpha_p(T)$ and $\prod_p \widetilde{u}_p$ are (absolutely or conditionally) convergent, and $\prod_p \widetilde{u}_p = L(1, \chi)$. This proves the assertion. □

For each $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ put $F_p^{(0)}(T; X) = F_p(p^{-e_p}T, X)$ and $\tilde{F}_p^{(0)}(T; X) = \tilde{F}_p(p^{-e_p}T, X)$. For a $GL_m(\mathcal{O}_p)$-invariant function ω_p on $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ put

$$\widetilde{P}_{m,p}(\omega_p, X, t) := \sum_{A \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/GL_m(\mathcal{O}_p)} \omega_p(A) t^{\mathrm{ord}_p(\det A)} \frac{\tilde{F}_p(A; X)}{\alpha_p(A)} .$$

Let $\iota_{m,p}$ be the constant function of $\widetilde{\mathrm{Her}}_{mp}^\times$ taking the value 1, and $\varepsilon_{m,p}$ the function of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ defined by

$$\varepsilon_{m,p}(A) = \begin{cases} 1 & \text{if } \det A \in N_{K_p/\mathbf{Q}_p}(K_p) \\ -1 & \text{otherwise} \end{cases} .$$

We sometimes drop the suffix and write $\iota_{m,p}$ as ι and the others if there is no fear of confusion. An explicit formula for $\widetilde{P}_{m,p}(\omega_p; X, t)$ will be given in the next section for $\omega_p = \iota_{m,p}$ and $\varepsilon_{m,p}$. Now put

$$\underline{\widetilde{\mathbf{Her}}_m} = \bigsqcup_{i=1}^{h} \widetilde{\mathrm{Her}}_m(\mathcal{O})_i^+ ,$$

where $t_i = (t_{i,p})$ is the element of $G^{(m)}(\mathbf{A}_f)$ defined before. We define the equivalence relation $\approx$ as follows: for two elements $A \in \prod_p (t_{i,p}^{*-1} \widetilde{\mathrm{Her}}_m(\mathcal{O}_p) t_{i,p}^{-1}) \cap \mathrm{Her}_m(K)$ and $B \in \prod_p (t_{j,p}^{*-1} \mathrm{Her}_m(\mathcal{O}_p) t_{j,p}^{-1}) \cap \mathrm{Her}_m(K)$ we write $A \approx B$ if $t_{i,p}^* A t_{i,p}$ is $GL_m(\mathcal{O}_p)$-equivalent to $t_{j,p}^* B t_{j.p}$ for any prime number p. We also put

$$\underline{\widetilde{\mathrm{Her}}_m'} = \{(A_p) \in \prod_p (\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/GL_m(\mathcal{O}_p) \mid \prod_p \varepsilon_p(A) = 1\} .$$

For each element $A \in \widetilde{\mathrm{Her}}_m(\mathcal{O})_i^+$ we put $\phi(A) = t_i^* A t_i$. Then $\phi(A)$ belongs to $\prod_p \mathrm{Her}_m(\mathcal{O}_p)$, and ϕ induces a mapping from $\underline{\widetilde{\mathbf{Her}}_m}/\approx$ to $\underline{\widetilde{\mathrm{Her}}_m'}$, which will be denoted also by ϕ.

PROPOSITION 3.2. *The above ϕ is bijective.*

Proof. The infectivity is clear. To prove the surjectivity, let (A_p) be an element of $\widetilde{\underline{\mathrm{Her}_m}}$. Then by the Hasse principle for Hermitian forms, there exists an element B' of $\overline{\mathrm{Her}_m}(K)$ such that

$$B' = \gamma_p^{*-1} A_p \gamma_p^{-1}$$

with some $\gamma_p \in GL_m(K_p)$ for any p. Then we may assume that $\gamma = (\gamma_p)$ is an element of $G(\mathbf{A}_f)$. Hence there exit an integer $1 \le i \le h$, $a \in GL_m(K)$, $(u_p) \in \prod_p GL_m(\mathcal{O}_p)$, and $u_\infty \in GL_m(\mathbf{C})$ such that

$$\gamma_p = a t_{i,p} u_p u_\infty$$

for any p. Put $B = a^* B' a$. Then it is easily seen that B belongs to $\underline{\widetilde{\mathbf{Her}}_m}$ and $\phi(B) = (A_p)$. This proves the assertion. $\square$

THEOREM 3.3. *For each $(A_p) \in \underline{\widetilde{\mathrm{Her}}'_m}$ and i, let $\mathcal{G}_i((A_p))$ denote the U_i-equivalence classes of elements T' of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_i)^+$ such that $t_{i,p}^* T' t_{i,p} \sim_{GL_m(\mathcal{O}_p)} A_p$ for any p. Moreover put*

$$M((A_p)) = \sum_{i=1}^{h} \sum_{T' \in \mathcal{G}_i((A_p))} \frac{1}{\#(U_{i,T'})}\,.$$

Then we have

$$M((A_p)) = \frac{2^{c_D m^2} \kappa_m}{D^{m(m+1)/4}} \prod_p \frac{p^{\mathrm{ord}_p(\det A_p)m}}{\alpha_p(A_p)}$$

Proof. By Proposition 3.2, there exist a positive integer $r \le h$ and an element $T \in \widetilde{\mathrm{Her}}_m(K)^+$ such that $t_{r,p}^* T t_{r,p} \sim_{GL_m(\mathcal{O}_p)} A_p$ for any p, and hence we have $M((A_p)) = M_r(T)$. Thus by Proposition 3.1, we prove the assertion. $\square$

THEOREM 3.4. *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $2n+1$. Put*

$$\mu_{m,k,D} = D^{m(s-k+l_0/2)+(k-l_0/2)[m/2]-m(m+1)/4}$$
$$\times 2^{-c_D m(s-k-2n-l_0/2)-m+1} \prod_{i=1}^{m} \Gamma_{\mathbf{C}}(i)\,,$$

where $l_0 = 0$ or 1 according as m is even or odd. Then for $\mathrm{Re}(s) >> 0$, *we have*

$$L^*(s, Lift^{(m)}(f)) = \frac{\mu_{m,k,D}}{2}$$
$$\times \left(\prod_p \widetilde{P}_{m,p}(\iota_p; \alpha_p^{-1}, p^{-s+k+2n+l_0/2}) + \prod_p \widetilde{P}_{m,p}(\varepsilon_p; \alpha_p^{-1}, p^{-s+k+2n+l_0/2}) \right)\,.$$

Proof. We have

$$L(s, Lift^{(m)}(f)) = \widetilde{D}^{ms} \sum_{i=1}^{h} \prod_p |\det t_{i,p}|_{K_p}^{s-k-n}$$

$$\times \sum_{T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_i)^+} (D^{[m/2]} \widetilde{D}^{-m} \det T)^{k-l_0/2} \prod_p |\det t_{i,p}|_{K_p}^{n+l_0/2} \widetilde{F}^{(0)}(t_{i,p}^* T t_{i,p}, \alpha_p^{-1}) .$$

Hence by (2) of Proposition 3.2, and Theorem 3.3, we have

$$L(s, Lift^{(m)}(f) = \widetilde{D}^{ms-m(k-l_0/2)} D^{[m/2](k-l_0/2)}$$

$$\times \sum_{(A_p) \in \underline{\widetilde{\mathrm{Her}}'_m}} \prod_p p^{(s+k-l_0/2)\mathrm{ord}_p(\det A_p)} \widetilde{F}^{(0)}(A_p, \alpha_p^{-1}) M((A_p))$$

$$= \mu_{m,k,D} \sum_{(A_p) \in \underline{\widetilde{\mathrm{Her}}'_m}} \prod_p \frac{p^{(s+k+2n+l_0/2)\mathrm{ord}_p(\det A_p)} \widetilde{F}^{(0)}(A_p, \alpha_p^{-1})}{\alpha_p(A_p)} .$$

Then by using the same argument as in the proof of Proposition 2.2 of [IS95], we prove the assertion. (See also [IK04].) □

By using the same argument as above, we can prove the following:

THEOREM 3.5. *Let $m = 2n$ or $2n+1$, and let $\mu_{m,k,D}$ and l_0 be as above. Then for* $\mathrm{Re}(s) >> 0$, *we have*

$$L^*(s, \mathbf{E}_{2k+2n}^{(m)}) = \frac{\mu_{m,k,D}}{2}$$

$$\times \left(\prod_p \widetilde{P}_{m,p}(\iota_p; p^{k-l_0/2}, p^{-s+k+2n+l_0/2}) + \prod_p \widetilde{P}_{m,p}(\varepsilon_p; p^{k-l_0/2}, p^{-s+k+2n+l_0/2}) \right) .$$

Now we consider a refined version of the mass formula for $\mathcal{SU}_T$. For an element $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})_i^+$, let $\mathcal{G}'_i(T)$ denote the set of U'_i-equivalence classes of positive definite Hermitian matrices $T' \in \mathrm{Her}_m(\mathcal{O})_i^+$, such that $t_i^* T' t_i$ is $SL_m(\mathcal{O}_p)$-equivalent to $t_i^* T t_i$ for any prime number p. Moreover put

$$M_i^*(T) = \sum_{T' \in \mathcal{G}'_i(T)} \frac{1}{\#(U'_i(T'))}$$

Let $\mathcal{U}_1$ be the unitary group defined in Section 1. Namely let

$$\mathcal{U}_1 = \{u \in \mathrm{Res}_{K/\mathbf{Q}}(GL_1) \mid \overline{u}u = 1\} .$$

For an element $T \in \mathrm{Her}_m(\mathcal{O}_p)$, let

$$\widetilde{U_{p,T}} = \{\det X \mid X \in \mathcal{U}_T(\mathcal{O}_p)\} .$$

Then $\widetilde{U_{p,T}}$ is a subgroup of $U_{1,p}$ of finite index. We then put $l_{p,T} = [U_{1,p} : \widetilde{U_{p,T}}]$. We also put

$$u_p = \begin{cases} (1+p^{-1})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ (1-p^{-1)^{-1}} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ 2^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

PROPOSITION 3.6. *Let $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})_r^+$. Then*

$$M_r^*(T) = \frac{2^{c_D m^2} \prod_p p^{\nu_{K_p}(\det t_{r,p})m} (\det T)^m \prod_{i=2}^m \Gamma_{\mathbf{C}}(i)}{2^{m-1} D^{m(m+1)/4+1/2} \prod_p u_p \mathfrak{l}_{p,t_r^* T t_r} \alpha_p(t_{r,p}^* T t_{r,p})},$$

where $c_D = 1$ or 0 according as 2 divides D or not.

Proof. The assertion can be proved by using the same argument as in the proof of Proposition 3.1 and [[Kat14], Proposition 3.1]. But for the sake of completeness we here give an outline of the proof. Let ω be the gauge form on $\mathcal{U}_T$ defined in the proof of proposition 3.1. We also define a gauge form $\widetilde{\omega}$ on $\mathcal{SU}_T$ by

$$\widetilde{\omega} = \omega/dt,$$

where dt is the 1-form on $\mathcal{U}_1$ normalized so that

$$\int_{\mathcal{U}_1(\mathcal{O}_p)} |dt|_p = u_p^{-1}.$$

Since $\mathcal{SU}_T$ is semi-simple, we can take the Tamagawa measure $|\widetilde{\omega}|_{\mathbf{A}}$ on $\mathcal{SU}_T$ as

$$|\widetilde{\omega}|_{\mathbf{A}} = |\widetilde{\omega}|_\infty \prod_{p<\infty} |\widetilde{\omega}|_p.$$

We recall that the Tamagawa number of $\mathcal{SU}_T$ is 1 (cf. Weil [We82]). We also note that the strong approximation theorem holds for SL_m, and hence we have

$$\mathcal{SU}_T(\mathbf{A}) = \mathcal{SU}_T(\mathbf{Q}) \mathcal{SK}_{r,T},$$

where

$$\mathcal{SK}_{r,T} = \prod_{p<\infty} (\mathcal{SU}_T(K_p) \cap t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1}).$$

Hence by using the standard method we can prove the following mass formula for the $\mathcal{SU}_T$:

$$M_r^*(T) \int_{\mathcal{SU}_T(\mathbf{R})} |\widetilde{\omega}|_\infty \prod_{p<\infty} \int_{\mathcal{SU}_T(K_p)\cap t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1}} |\widetilde{\omega}|_p = 1.$$

We have

$$\int_{\mathcal{SU}_T(\mathbf{R})} |\widetilde{\omega}|_\infty = \left(\frac{(\det T)^m D^{(m(m+1)-2)/4} \Gamma_{\mathbf{C}}(i)}{2^{m-1}} \right)^{-1}.$$

On the other hand, we have

$$\upsilon_p(t_{r,p}^* T t_{r,p}) = p^{\nu_{K_p}(\det t_{r,p})m} \int_{\mathcal{U}_T(K_p)\cap t_{r,p} GL_m(\mathcal{O}_p) t_{r,p}^{-1}} |\omega|_p$$

$$= \int_{\mathcal{SU}_T(K_p)\cap t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1}} |\widetilde{\omega}|_p \int_{\widetilde{U_{p,T}}} |dt|_p = u_p^{-1} l_{p,t_r^* T t_r}^{-1} \int_{\mathcal{SU}_T(K_p)\cap t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1}} |\widetilde{\omega}|_p.$$

This completes the proof. □

For a subset $\mathcal{T}$ of K_p put

$$\mathrm{Her}_m(\mathcal{T}) = \mathrm{Her}_m(K_p) \cap M_m(\mathcal{T}),$$

and for a subset $\mathcal{S}$ of K_p put

$$\mathrm{Her}_m(\mathcal{S}, \mathcal{T}) = \{A \in \mathrm{Her}_m(\mathcal{T}) \mid \det A \in \mathcal{S}\},$$

and $\widetilde{\mathrm{Her}}_m(\mathcal{S}, \mathcal{T}) = \mathrm{Her}_m(\mathcal{S}, \mathcal{T}) \cap \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. In particular if $\mathcal{S}$ consists of a single element d we write $\mathrm{Her}_m(\mathcal{S}, \mathcal{T})$ as $\mathrm{Her}_m(d, \mathcal{T})$, and so on. For $d \in \mathbf{Z}_{>0}$ we also define the set $\mathrm{Her}_m(d, \mathcal{O})^+$ in a similar way. Now let

$$\widetilde{\mathsf{Her}}_m = \prod_p (\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/SL_m(\mathcal{O}_p)).$$

For $T \in \mathrm{Her}_m(\mathcal{O})_r^+$, let $\phi_r(T)$ be the diagonal embedding of $t_r^* T t_r$ into $\prod_p \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. The mapping $T \mapsto \phi_r(T)$ induces a mapping from $\widetilde{\mathrm{Her}}_m(\mathcal{O})_r^+ / \prod_p (t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1})$ to $\widetilde{\mathsf{Her}}_m$, which will be denoted also by ϕ_r.

For a positive rational number d such that $\mathrm{ord}_p(d) \geq -\mathrm{ord}_p(\det t_{r,p} \overline{\det t_{r,p}})$ for any prime number p, we put

$$\widetilde{\mathrm{Her}}_m(d, \mathcal{O})_r^+ = \widetilde{\mathrm{Her}}_m(\mathcal{O})_r^+ \cap \mathrm{Her}_m(d, K),$$

and

$$\widetilde{\mathsf{Her}}_m(d)_r = \prod_p (\widetilde{\mathrm{Her}}_m(d N_{K_p/\mathbf{Q}_p}(\det t_{r,p}), \mathcal{O}_p)/SL_m(\mathcal{O}_p)).$$

PROPOSITION 3.7. *Let the notation and the assumption be as above. Then the mapping ϕ_r induces a bijection from $\widetilde{\mathrm{Her}}_m(d, \mathcal{O})_r^+ / \prod_p (t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1})$ to $\widetilde{\mathsf{Her}}_m(d)_r$, which will be denoted also by ϕ_r.*

Proof. Put $RSL_m = \mathrm{Res}_{K/\mathbf{Q}} SL_m$. Then by the strong approximation theorem for SL_m we have

$$RSL_{m,\mathbf{A}} = RSL_m(\mathbf{Q}) (\prod_p t_{r,p} SL_m(\mathcal{O}_p) t_{r,p}^{-1}) RSL_m(\mathbf{R}).$$

Then the assertion can be proved by using the same argument as in the proof of [[Kat14], Proposition 3.3]. □

For $d \in \mathbf{Z}_p^\times$, put

$$\lambda_{m,p}(d, X) = \sum_{A \in \widetilde{\mathrm{Her}}_m(d, \mathcal{O}_p)/SL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A, X)}{u_p l_{p,A} \alpha_p(A)}.$$

THEOREM 3.8. *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $2n+1$. For an integer $1 \leq r \leq h$ let $c_r = \prod_p p^{-\nu_{K_p}(\det t_{r,p})}$ be as before, and a positive integer d_0 put*

$$b_{m,r}(f; d_0) = \prod_p \lambda_{m,p}(c_r N_{K_p/\mathbf{Q}_p}(\det t_{r,p}) d_0, \alpha_p^{-1}),$$

where α_p is the Satake p-parameter of f. Then for $\mathrm{Re}(s) >> 0$, *we have*

$$L(s, I_m(f)_r) = \mu_{m,k,D} \sum_{d_0=1}^{\infty} b_{m,r}(f; d_0) d_0^{-s+k+2n+l_0/2}.$$

Proof. The assertion can be proved by Propositions 3.6 and 3.7 in the same way as [[Kat14], Theorem 3.4]. □

4. Proof of Theorem 2.1

For $d_0 \in \mathbf{Z}_p^{\times}$ put

$$\hat{P}_{m,p}(d_0, X, t) = \sum_{i=0}^{\infty} \lambda_{m,p}^{*}(p^i d_0, X) t^i ,$$

where for $d \in \mathbf{Z}_p^{\times}$ we define $\lambda_{m,p}^{*}(p^i d_0, X)$ as

$$\lambda_{m,p}^{*}(d, X) = \sum_{A \in \widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*), \mathcal{O}_p)/GL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A, X)}{\alpha_p(A)} .$$

We note that

$$\sum_{A \in \widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*), \mathcal{O}_p)/GL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A, X)}{\alpha_p(A)}$$

is $\chi_{K_p}((-1)^{m/2} d) \lambda_{m,p}^{*}(d, X)$ or $\lambda_{m,p}^{*}(d, X)$ according as m is even and K_p is a field, or not.

Remark 1. By [[Kat14], Proposition 4.3.7], we have

$$\lambda_{m,p}^{*}(d, X) = u_p \lambda_{m,p}(d, X)$$

for $d \in \mathbf{Z}_p^{\times}$ and therefore

$$\hat{P}_{m,p}(d_0, X, t) = u_p \sum_{i=0}^{\infty} \lambda_{m,p}(p^i d_0, X) t^i .$$

We also define $P_{m,p}(d_0, X, t)$ as

$$P_{m,p}(d_0, X, t) = \sum_{i=0}^{\infty} \lambda_{m,p}^{*}(\pi_p^i d_0, X) t^i .$$

Let $m = 2n$ be even and suppose that K_p is ramified over $\mathbf{Q}_p$. For $l = 0, 1$ put

$$P_{2n,p}^{(l)}(X, t) = \frac{1}{2} \sum_{d \in \mathcal{N}_p} \chi_{K_p}((-1)^n d)^l P_{2n,p}(d, X, t) ,$$

and

$$\hat{P}_{2n,p}^{(l)}(X, t) = \frac{1}{2} \sum_{d \in \mathcal{N}_p} \chi_{K_p}((-1)^n d)^l \hat{P}_{2n,p}(d, X, t) .$$

Remark 2. We note that $P_{m,p}(d_0, X, t) = \hat{P}_{m,p}(d_0, X, t)$ unless m is even and K_p is ramified over $\mathbf{Q}_p$. Let $m = 2n$ be even and suppose that K_p is ramified over $\mathbf{Q}_p$. Then we have

$$\hat{P}^{(0)}_{2n,p}(X, t) = P^{(0)}_{2n,p}(X, t) \quad \text{and } \hat{P}^{(1)}_{2n,p}(X, t) = P^{(1)}_{2n,p}(X, \chi_{K_p}(p)t)$$

(cf. [[Kat14], Corollary to Theorem 4.3.6].)

Proof of (1) of Theorem 2.3 The proof is merely a modification of the proof of [[Kat14], Theorem 2.3]. But for the sake of convenience, we here give a complete proof. For a while put $\lambda^*_p(d) = \lambda^*_{m,p}(d, \alpha_p^{-1})$. Then by Theorem 3.8 and Remark 1, we have

$$L(s, I_{2n}(f)_r) = \mu_{2n,k,D} \sum_{d=1}^{\infty} \prod_p (u_p^{-1} \lambda^*_p(c_r N_{K_p/\mathbf{Q}_p}(\det t_{r,p})d)) d^{-s+k+2n} .$$

Then by (1) and (2) of Theorem 4.3.1, and (1) of [[Kat14], Theorem 4.3.6], $\lambda^*_p(c_r N_{K_p/\mathbf{Q}_p}(\det t_{r,p})d)$ depends only on $p^{\mathrm{ord}_p(d)}$ if $p \nmid D$. Hence we write $\lambda^*_p(c_r N_{K_p/\mathbf{Q}_p}(\det t_{r,p})d)$ as $\widetilde{\lambda}_p(p^{\mathrm{ord}_p(d)})$. On the other hand, if $p \mid D$, by (3) of [[Kat14], Theorem 4.3]. and , (2) of [[Kat14], Theorem 4.3.6], $\lambda^*_p(d)$ can be expressed as

$$\lambda^*_p(d) = \lambda^{(0)}_p(d) + \chi_{K_p}((-1)^n c_r N_{K_p/\mathbf{Q}_p}(\det t_{r,p}) d p^{-\mathrm{ord}_p(d)}) \lambda^{(1)}_p(d) ,$$

where $\lambda^{(l)}_p(d)$ is a rational number depending only on $p^{\mathrm{ord}_p(d)}$ for $l = 0, 1$. Hence we write $\lambda^{(l)}_p(d)$ as $\widetilde{\lambda}^{(l)}_p(p^{\mathrm{ord}_p(d)})$. Note that $\chi_{K_p}((-1)^n c_r N_{K_p/\mathbf{Q}_p}(\det t_{r,p}) d p^{-\mathrm{ord}_p(d)}) = \chi_{K_p}((-1)^n c_r d p^{-\mathrm{ord}_p(d)})$. Hence we have

$$\begin{aligned} b_{m,r}(f; d) = \sum_{Q \in Q_D} & \prod_{p \mid d, p \nmid D} \left(u_p^{-1} \widetilde{\lambda}_p(p^{\mathrm{ord}_p(d)}) \prod_{q \in Q} \chi_{K_q}(p^{\mathrm{ord}_p(d)}) \right) \\ & \times \prod_{p \mid d, p \mid D, p \notin Q} \left(u_p^{-1} \widetilde{\lambda}^{(0)}_p(p^{\mathrm{ord}_p(d)}) \prod_{q \in Q} \chi_{K_q}(p^{\mathrm{ord}_p(d)}) \right) \\ \times \prod_{p \mid d, p \in Q} & \left(u_p^{-1} \widetilde{\lambda}^{(1)}_p(q^{\mathrm{ord}_p(d)}) \prod_{q \in Q, q \neq p} \chi_{K_p}(p^{\mathrm{ord}_p(d)}) \right) \prod_{q \in Q} \chi_{K_q}((-1)^n c_r)) \end{aligned}$$

for a positive integer d. We note that for a subset Q of Q_D we have

$$\chi_Q(m) = \prod_{q \in Q} \chi_{K_q}(m)$$

for an integer m coprime to any $q \in Q$, and

$$\chi'_Q(p) = \chi_{K_p}(p) \prod_{q \in Q, q \neq p} \chi_{K_q}(p)$$

for any $p \in Q$. Hence, by [[Kat14], Theorems 4.3.1 and 4.3.6], and Remark 2, we have

$$L(s, I_{2n}(f)_r) = \mu_{2n,k,D} \sum_{Q \subset Q_D} \prod_{p \nmid D} \sum_{i=0}^{\infty} u_p^{-1} \widetilde{\lambda}_p(p^i) \chi_Q(p^i) p^{(-s+k+2n)i}$$

$$\times \prod_{p|D, p\not\in Q} \sum_{i=0}^{\infty} u_p^{-1}\widetilde{\lambda}_p^{(0)}(p^i)\chi_Q(p^i)p^{(-s+k+2n)i}\chi_Q((-1)^n c_r)$$

$$\times \prod_{p\in Q} \sum_{i=0}^{\infty} u_p^{-1}\widetilde{\lambda}_p^{(1)}(p^i)\left(\prod_{q\in Q, q\neq p} \chi_{K_q}(p^i)\right) p^{(-s+k+2n)i}.$$

$$= \mu_{2n,k,D} \sum_{Q\subset Q_D} \chi_Q((-1)^n c_r) \prod_{p\nmid D} (u_p^{-1} P_{2n,p}(1, \alpha_p^{-1}, \chi_Q(p)p^{-s+k+2n}))$$

$$\times \prod_{p|D, p\not\in Q} (u_p^{-1} P_{2n,p}^{(0)}(\alpha_p^{-1}, \chi_Q(p)p^{-s+k+2n})) \prod_{p\in Q} (u_p^{-1} P_{2n,p}^{(1)}(\alpha_p^{-1}, \chi_Q'(p)p^{-s+k+2n}))\,.$$

Now for $l = 0, 1$ write $P_{2n,p}^{(l)}(X, t)$ as

$$P_{2n,p}^{(l)}(X, t) = t^{ni_p} \widetilde{P}_{2n,p}^{(l)}(X, t)\,,$$

where $i_p = 0$ or 1 according as $4||D$ and $p = 2$, or not. Notice that $u_p = (1-\chi(p)p^{-1})^{-1}$ if $p \nmid D$ and $u_p = 2^{-1}$ if $p|D$. Hence we have

$$L(s, I_{2n}(f)_r) = \mu_{2n,k,D} \sum_{Q\subset Q_D} \chi_Q((-1)^n c_r)$$

$$\times \prod_{p\in Q_D'} p^{(-s+k+2n)n}\Big(\prod_{p\in Q_D, p\not\in Q} \chi_Q(p) \prod_{p\in Q} \chi_Q'(p)\Big)^n$$

$$\times \prod_{p\nmid D} ((1-\chi(p)p^{-1})P_{2n,p}(1, \alpha_p^{-1}, \chi_Q(p)p^{-s+k+2n}))$$

$$\times \prod_{p|D, p\not\in Q} (2\widetilde{P}_{2n,p}^{(0)}(\alpha_p^{-1}, \chi_Q(p)p^{-s+k+2n})) \prod_{p\in Q} (2\widetilde{P}_{2n,p}^{(1)}(\alpha_p^{-1}, \chi_Q'(p)p^{-s+k+2n}))\,,$$

where $Q_D' = Q_D \backslash \{2\}$ or Q_D according as $4||D$ or not. Note that

$$2^{2c_D n(-s+k+2n)} \prod_{p\in Q_D'} p^{(-s+k+2n)n} = D^{(-s+k+2n)n}\,,$$

and

$$\prod_{p\in Q_D, p\not\in Q} \chi_Q(p) \prod_{p\in Q} \chi_Q'(p) = 1.$$

Thus the assertion follows from [[Kat14], Theorem 4.3.1].

Proof of (2) of Theorem 2.1. The assertion follows directly from Theorem 3.8 and [[Kat14], Theorems 4.3.2].

5. Another proof of Theorem 2.2

In this section, we give another proof to Theorem 2.2. This is much simpler than the previous proof, and it seems interesting in its own right. This method is also applicable to a proof of Theorem 2.3. To do this, we give the relation between two formal power series $\hat{P}_{m,p}(d_0, X, t)$ and $\widetilde{P}_{m,p}(\varepsilon^l, X, t)$.

PROPOSITION 5.1. *Let m be a positive integer.*
(1) *Assume that K_p is an unramified quadratic extension of $\mathbf{Q}_p$. Then we have*

$$\widetilde{P}_{m,p}(\iota, X, t) = \hat{P}_{m,p}(d_0, X, t)\,,$$

and

$$\widetilde{P}_{m,p}(\varepsilon, X, t) = \hat{P}_{m,p}(d_0, X, -t)$$

for any $d_0 \in \mathbf{Z}_p^*$.

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then we have*

$$\widetilde{P}_{m,p}(\iota, X, t) = \widetilde{P}_{m,p}(\varepsilon, X, t) = \hat{P}_{m,p}(d_0, X, t)\,,$$

for any $d_0 \in \mathbf{Z}_p^*$.

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then we have*

$$\widetilde{P}_{m,p}(\iota, X, t) = \sum_{d \in \mathcal{N}_p} \hat{P}_{m,p}(d, X, t)\,,$$

and

$$\widetilde{P}_{m,p}(\varepsilon, X, t) = \sum_{d \in \mathcal{N}_p} \chi_{K_p}(d) \hat{P}_{m,p}(d, X, t)\,.$$

Put $\phi_m(q) = \prod_{i=1}^m (1 - q^i)$.

By explicit formulas for $\hat{P}_{m,p}(d, X, t)$ in [[Kat14], Theorem 4.3.6], we have the following two theorems:

THEOREM 5.2. *Let $m = 2n$ be even.*
(1) *Assume that K_p is an unramified quadratic extension of $\mathbf{Q}_p$. Then*

$$\widetilde{P}_{2n,p}(\iota, X, t) = \frac{1}{\phi_m(-p^{-1}) \prod_{i=1}^m (1 - t(-p)^{-i} X)(1 + t(-p)^{-i} X^{-1})}\,,$$

and

$$\widetilde{P}_{2n,p}(\varepsilon, X, t) = \frac{1}{\phi_m(-p^{-1}) \prod_{i=1}^m (1 + t(-p)^{-i} X)(1 - t(-p)^{-i} X^{-1})}\,.$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for $\omega = \iota, \varepsilon$,*

$$\widetilde{P}_{2n,p}(\omega, X, t) = \frac{1}{\phi_m(p^{-1}) \prod_{i=1}^m (1 - tp^{-i} X)(1 - tp^{-i} X^{-1})}\,.$$

(3) *Assume that K_p is a ramified quadratic extension of $\mathbf{Q}_p$. Then*

$$\widetilde{P}_{2n,p}(\iota, X, t) = \frac{1}{\phi_n(p^{-2})} \frac{t^{ni_p}}{\prod_{i=1}^n (1 - tp^{-2i+1} X^{-1})(1 - tp^{-2i} X)}\,,$$

and

$$\widetilde{P}_{2n,p}(\varepsilon, X, t) = \frac{1}{\phi_n(p^{-2})} \frac{\chi_{K_p}((-1)^n d_0)(t\chi_{K_p}(p))^{ni_p}}{\prod_{i=1}^n (1 - tp^{-2i}\chi_{K_p}(p)X^{-1})(1 - tp^{-2i+1}\chi_{K_p}(p)X)}.$$

THEOREM 5.3. *Let $m = 2n+1$ be odd.* (1) *Assume that K_p is unramified over $\mathbf{Q}_p$. Then*

$$\widetilde{P}_{2n+1,p}(\iota, X, t) = \frac{1}{\phi_m(-p^{-1}) \prod_{i=1}^m (1 + t(-p)^{-i}X)(1 + t(-p)^{-i}X^{-1})},$$

and

$$\widetilde{P}_{2n+1,p}(\varepsilon, X, t) = \frac{1}{\phi_m(-p^{-1}) \prod_{i=1}^m (1 - t(-p)^{-i}X)(1 - t(-p)^{-i}X^{-1})}.$$

(2) *Assume that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for $\omega = \iota,\ \varepsilon$.*

$$\widetilde{P}_{2n+1,p}(\omega, X, t) = \frac{1}{\phi_m(p^{-1}) \prod_{i=1}^m (1 - tp^{-i}X)(1 - tp^{-i}X^{-1})}.$$

(3) *Assume that K_p is ramified over $\mathbf{Q}_p$. Then*

$$\widetilde{P}_{2n+1,p}(\iota, X, t) = \frac{t^{(m+1)i_p/2+\delta_{2p}}}{\phi_{(m-1)/2}(p^{-2}) \prod_{i=1}^{(m+1)/2} (1 - tp^{-2i+1}X)(1 - tp^{-2i+1}X^{-1})},$$

and

$$\widetilde{P}_{2n+1,p}(\varepsilon, X, t) = 0.$$

Proof of Theorems 2.2 and 2.3. Theorems 2.2 and 2.3 follow directly from Theorems 3.4, 3.5, 5.2, and 5.3.

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